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Two-Dimensional Noncommutative Quantum Dynamics

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This paper gives the two-dimensional extension for the noncommutative quantum dynamics of Rembielinski and Smolinski.

Since the introduction of the noncommutative plane (so-called quantum plane) (Wess and Zumino, 1990) many theoretical physicists have attempted to build physical models based on this type of noncommutative geometry. The work of Aref'eva and Volovich (1991), Schwenk and Wess (1992), and Rembielinski and Smolinski (1993) relates to one-dimensional quantum dynamics only. In the commutative plane we can easily extend the results obtained in one dimension to more general cases, those in N dimensions. However, this is not the case in the quantum plane. In this paper we extend the result given in Rembielinski and Smolinski (1993) to the two-dimensional case.

Our starting point for the noncommutative quantum dynamics in two dimensions is the following extended Hamiltonian:

$$H = p_x^2 K_x^2 + p_y^2 K_y^2 + V(x, K_x, \Lambda_x, y, K_y, \Lambda_y) \tag{1}$$

The commutation relations between coordinates and momenta are given by

$$xp_x = q^2 p_x x + i\hbar q \Lambda_x^2 + \lambda y p_y$$

$$yp_x = qp_x y$$

$$xp_y = qp_y x$$

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$$y p_y = q^2 p_y y + i\hbar \Lambda_y^2$$
$$p_x p_y = q p_y p_x$$
$$xy = q^{-1} yx \qquad (2)$$

where $\lambda = q^2 - 1$ and K_x, K_y, Λ_x, and Λ_y are assumed to be additional Hermitian generators of the extended noncommutative algebra in two dimensions. Here we assume that the commutation relations among coordinates, momenta, and additional generators take the following form:

$$x\Lambda_x = \xi_x \Lambda_x x$$
$$p_x \Lambda_x = \xi_x^{-1} \Lambda_x p_x$$
$$y\Lambda_y = \xi_y \Lambda_y y$$
$$p_y \Lambda_y = \xi_y^{-1} \Lambda_y p_y$$
$$x\Lambda_y = \xi_1 \Lambda_y x$$
$$y\Lambda_x = \xi_2 \Lambda_x y$$
$$p_x \Lambda_y = \xi_3 \Lambda_y p_x$$
$$p_y \Lambda_x = \xi_4 \Lambda_x p_y$$
$$\Lambda_x \Lambda_y = \eta \Lambda_y \Lambda_x$$
$$xK_x = \tau_x^2 K_x x$$
$$p_x K_x = \epsilon_x^2 K_x p_x$$
$$yK_y = \tau_y^2 K_y y$$
$$p_y K_y = \epsilon_y^2 K_y p_y$$
$$xK_y = \tau_1 K_y x$$
$$yK_x = \tau_2 K_x y$$
$$p_x K_y = \tau_3 K_y p_x$$
$$p_y K_x = \tau_4 K_x p_y$$
$$\Lambda_x K_x = \eta_1 K_x \Lambda_x$$
$$\Lambda_x K_y = \eta_2 K_y \Lambda_x$$
$$\Lambda_y K_y = \eta_3 K_y \Lambda_y$$
$$\Lambda_y K_x = \eta_4 K_x \Lambda_y \qquad (3)$$

The consistency condition of this system requires

$$
\begin{aligned}
\xi_4^2\xi_2^2 &= 1, & \eta^{-2} &= \xi_2\xi_4 \\
\xi_1\xi_3 &= 1, & \eta &= 1 \\
\eta_1 &= \epsilon_x\tau_x, & \tau_2\tau_4 &= (\epsilon_x\tau_x)^2 \\
\eta_2 &= \epsilon_y\tau_y, & \tau_1\tau_3 &= (\tau_y\epsilon_y)^2 \\
\eta_3 &= \epsilon_y\tau_y & \eta_4 &= (\tau_2\tau_4)^{1/2} = \epsilon_x\tau_x
\end{aligned}
$$

If we assume that Λ_x, Λ_y, K_x, and K_y are constant in time, we have

$$
\begin{aligned}
\dot{\Lambda}_x &= \frac{i}{\hbar}\,[H, \Lambda_x] = 0 \\
\dot{\Lambda}_y &= \frac{i}{\hbar}\,[H, \Lambda_y] = 0 \\
\dot{K}_x &= \frac{i}{\hbar}\,[H, K_x] = 0 \\
\dot{K}_y &= \frac{i}{\hbar}\,[H, K_y] = 0
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\epsilon_x\tau_x\xi_x &= 1, & \epsilon_y\tau_y\xi_2 &= 1 \\
\epsilon_x\tau_x\xi_1 &= 1, & \epsilon_y\tau_y\xi_y &= 1 \\
\epsilon_x &= 1, & \tau_4 &= 1 \\
\epsilon_y &= 1, & \tau_3 &= 1 \\
\tau_x &= \xi_x^{-1}, & \tau_y &= \xi_2^{-1} \\
\tau_x &= \xi_1^{-1}, & \tau_y &= \xi_y^{-1} \\
\xi_1 &= \xi_x, & \xi_2 &= \xi_y \\
\tau_1 &= \xi_y^{-2}, & \tau_2 &= \xi_x^{-2}
\end{aligned}
$$

and the potential energy should satisfy

$$
\begin{aligned}
&V(\xi_x x, \xi_x K_x, \Lambda_x, \xi_y y, \xi_y K_y, \Lambda_y) \\
&\quad = V(\xi_x^2 x, K_x, \xi_x\Lambda_x, \xi_x^2 y, K_y, \xi_x\Lambda_y) \\
&\quad = V(\xi_y^2 x, K_x, \xi_y\Lambda_x, \xi_y^2 y, K_y, \xi_y\Lambda_y) \\
&\quad = V(x, K_x, \Lambda_x, y, K_y, \Lambda_y) \qquad (5)
\end{aligned}
$$

Then the extended noncommutative relations among coordinates, momenta, and additional generators take the following form:

$$\begin{aligned}
x\Lambda_x &= \xi_x\Lambda_x x \\
p_x\Lambda_x &= \xi_x^{-1}\Lambda_x p_x \\
y\Lambda_y &= \xi_y\Lambda_y y \\
p_y\Lambda_y &= \xi_y^{-1}\Lambda_y p_y \\
x\Lambda_y &= \xi_x\Lambda_y x \\
y\Lambda_x &= \xi_y\Lambda_x y \\
p_x\Lambda_y &= \xi_x^{-1}\Lambda_y p_x \\
p_y\Lambda_x &= \xi_y^{-1}\Lambda_x p_y \\
\Lambda_x\Lambda_y &= \Lambda_y\Lambda_x \\
xK_x &= \xi_x^{-2}K_x x \\
p_xK_x &= K_x p_x \\
yK_y &= \xi_y^{-2}K_y y \\
p_yK_y &= K_y p_y \\
xK_y &= \xi_y^{-2}K_y x \\
yK_x &= \xi_x^{-2}K_x y \\
p_xK_y &= K_y p_x \\
p_yK_x &= K_x p_y \\
\Lambda_xK_x &= \xi_x^{-1}K_x\Lambda_x \\
\Lambda_xK_y &= \xi_y^{-1}K_y\Lambda_x \\
\Lambda_yK_y &= \xi_y^{-1}K_y\Lambda_y \\
\Lambda_yK_x &= \xi_x^{-1}K_x\Lambda_y
\end{aligned} \tag{6}$$

Then the Heisenberg equations of motion for x and y are given by

$$\begin{aligned}
\dot{x} &= \frac{i}{\hbar}[H, x] \\
&= \left[\frac{i}{\hbar}(\xi_x^4 - q_4)p_x^2x + q(q^2 + \xi_x^2)p_x\Lambda_x^2 + i\hbar\lambda(q^2 + 1)p_xyp_y\right]K_x^2 \\
&\quad + \frac{i}{\hbar}(\xi_y^2 - q^2)p_y^2xK_y^2
\end{aligned} \tag{7}$$

and

$$\begin{aligned}\dot{y} &= \frac{i}{\hbar}[H, y] \\ &= \frac{i}{\hbar}(\xi_x^2 - q^2)p_x^2 y K_x^2 \\ &\quad + \left[\frac{i}{\hbar}(\xi_x^4 - q^4)p_y^2 y + q(q^2 + \xi_y^2)p_y \Lambda_y^2\right]K_y^2 \end{aligned} \tag{8}$$

where we set

$$q = \left(\frac{\xi_y}{\xi_x}\right)^2$$

in order to make the potential terms vanish. Similarly, we have the Heisenberg equation of motion for the momenta p_x and p_y:

$$\begin{aligned}\dot{p}_x &= \frac{i}{\hbar}[H, p_x] \\ &= \frac{i}{\hbar}(1 - q^2)p_y^2 p_x K_y^2 \\ &\quad + \frac{i}{\hbar}p_x[V(q^2 x, K_x, \Lambda_x, qy, K_y, \Lambda_y) - V(x, K_x, \Lambda_x, y, K_y, \Lambda_y)] \\ &\quad - q\frac{d}{d_{(q/\xi_x)^2 x}}V(x, K_x, \Lambda_x, qy, K_y, \Lambda_y) \\ &\quad + \frac{i}{\hbar}\lambda q p_y \frac{d}{d_q x}V(qx, K_x, \xi_y\Lambda_x, qy, K_y, \xi_y\Lambda_y) \\ &\quad - q\Lambda_y^2\frac{d}{d_q x}V(x, \xi_x^2 K_x, \Lambda_x, qy, \xi_y^2 K_y, \Lambda_y)\end{aligned} \tag{9}$$

and

$$\begin{aligned}\dot{p}_y &= \frac{i}{\hbar}[H, p_y] \\ &= \frac{i}{\hbar}(q^2 - 1)p_y p_x^2 K_x^2 \\ &\quad + \frac{i}{\hbar}p_y[V(qx, K_x, \xi_y\Lambda_x, q^2 y, K_y, \xi_y\Lambda_y) - V(x, K_x, \Lambda_x, y, K_y, \Lambda_y)] \\ &\quad - q\frac{d}{d_{(q/\xi_x)^2 y}}V(qx, \xi_x^2 K_x, \Lambda_x, \xi_y y, \xi_y^{-2} K_y, \Lambda_y)\end{aligned} \tag{10}$$

where

$$\frac{d}{d_k x} V(x, K_x, \Lambda_x, y, K_y, \Lambda_y)$$
$$= \frac{V(kx, K_x, \Lambda_x, y, K_y, \Lambda_y) - V(x, K_x, \Lambda_x, y, K_y, \Lambda_y)}{x(k-1)}$$

$$\frac{d}{d_k y} V(x, K_x, \Lambda_x, y, K_y, \Lambda_y)$$
$$= \frac{V(x, K_x, \Lambda_x, ky, K_y, \Lambda_y) - V(x, K_x, \Lambda_x, y, K_y, \Lambda_y)}{y(k-1)}$$

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